

## BRIEF COMMUNICATION

### ON ONE-DIMENSIONAL MOTION OF A SINGLE COMPONENT IN TWO PHASES\*

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#### INTRODUCTION

Many discussions have been contributed to the subject of mathematical modeling of two-phase flow (e.g. Gidaspow 1974). General formulations of multiphase systems have been presented (e.g. Soo 1967); however, the interest in generality tends to obscure the essentials for describing the interactions in two-phase systems. It appears desirable to examine the simple case of one-dimensional adiabatic motion of a single component, two-phase system such as a steam-water mixture. Interactions with the boundary are not included here for the sake of simplicity. Flow regimes may include suspensions of droplets or bubbles, slugs, or layers with identifiable average characteristics.

#### INTRAPHASE RELATIONS

Within each phase (1, 2), the continuity equation can be written as

$$\frac{\partial \bar{\rho}_1}{\partial t} + \frac{\partial \bar{\rho}_1 u_1}{\partial x} = \bar{\Gamma}_1 \quad [1]$$

where  $t$  is time;  $x$ , the space coordinate;  $\bar{\rho}_1$ ,  $\bar{\Gamma}_1$ , density of material constituting phase 1 and the source (or sink) in that phase; and  $u_1$ , the mean velocity of phase 1. The momentum equation of phase 1 is then

$$\bar{\rho}_1 \frac{du_1}{dt_1} = \bar{\rho}_1 \frac{\partial u_1}{\partial t} + \bar{\rho}_1 u_1 \frac{\partial u_1}{\partial x} = -\frac{\partial P}{\partial x} + \bar{I}_{12} + \bar{V}_{12} \quad [2]$$

where  $P$  is the pressure of the system;  $\bar{I}_{12}$ , the inertial interaction force exerted on phase 1 by phase 2; and  $\bar{V}_{12}$ , the corresponding viscous interaction force.  $\bar{I}_{12}$  includes the effect of phase change (source or sink) and that of virtual mass. Validity of [1] and [2] calls for consideration of a continuous domain of phase 1, which may consist of slugs or droplets; the details can be accounted for in the boundary conditions.

Equations [1] and [2] are applicable to all the regimes of the mixture of phases; for example,  $P$  is the pressure in each phase in annular flow and is the pressure in the vapor bubble and the inside of the liquid droplets in a droplet suspension. For the sake of simplicity, field forces and capillarity are neglected in the present formulation. It is understood that correction may be needed when small radius of curvature exists.

#### INTERPHASE RELATIONS

When converted to treating the behaviors of a phase in a mixture of volume fractions  $\phi_1$ ,  $\phi_2$  of the phases ( $\phi_1 + \phi_2 = 1$ ), the density of phase 1 in the mixture is  $\rho_1 = \phi_1 \bar{\rho}_1$ , [2] can be rewritten as:

$$\rho_1 \frac{du_1}{dt_1} = \rho_1 \frac{\partial u_1}{\partial t} + \rho_1 u_1 \frac{\partial u_1}{\partial x} = -\phi_1 \frac{\partial P}{\partial x} + I_{12} + V_{12} \quad [3]$$

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and

$$I_{12} = \phi_1 \bar{I}_{12}, \quad V_{12} = \phi_1 \bar{V}_{12}. \quad [4]$$

Therefore, consistent treatment of all two-phase systems requires that each phase is subject to its partial gradient of pressure ( $\phi_1 \partial P / \partial x$ ) rather than gradient of partial pressure ( $P_1 = \phi_1 P$ ). This condition was not obvious when the derivation proceeded from a homogeneous mixture alone (Soo 1967).

The continuity equation of phase 1 in the mixture is given by

$$\frac{\partial \rho_1}{\partial t} + \frac{\partial \rho_1 u_1}{\partial x} = \Gamma_1 \quad [5]$$

where  $\Gamma_1$  is the generation rate of phase 1 per unit volume of the mixture, and for the mixture, where  $\Gamma_1 + \Gamma_2 = 0$  and we shall denote  $\Gamma_1 = -\Gamma_2 = \Gamma$ . Comparing [5] to [1] gives

$$\bar{\Gamma}_1 = \frac{\Gamma}{\phi_1} - \frac{\bar{\rho}_1}{\phi_1} \frac{d\phi_1}{dt_1}. \quad [6]$$

Hence,  $\bar{\Gamma}_1$  includes an apparent source due to variation in  $\phi_1$ . Adding  $\bar{\Gamma}_1$  to a corresponding relation for  $\bar{\Gamma}_2$  shows that  $\bar{\Gamma}_1 + \bar{\Gamma}_2 \neq 0$ .

#### INTERACTION FORCES

The above quantities ( $I$ 's and  $V$ 's) can be determined from considering the basic relations of the mixture (subscript  $m$ ) by

$$\frac{\partial \rho_m}{\partial t} + \frac{\partial \rho_m u_m}{\partial x} = 0, \quad [7]$$

$$\rho_m \frac{du_m}{dt_m} = \rho_m \frac{\partial u_m}{\partial t} + \rho_m u_m \frac{\partial u_m}{\partial x} = -\frac{\partial P}{\partial x} \quad [8]$$

where  $\rho_m = \rho_1 + \rho_2$  and  $\rho_m u_m = \rho_1 u_1 + \rho_2 u_2$ . The virtual mass effect does not enter the overall momentum equation because action and reaction of the phases cancel. Comparison to [3] and the corresponding relation for phase 2 gives the inertial interaction forces per unit volume of the mixture:

$$I_{12} = \frac{\rho_1}{\bar{\rho}_1} \bar{I}_{12} = -\frac{\rho_2}{\rho_1 + \rho_2} (u_1 - u_2) \Gamma + \frac{\partial}{\partial x} \left[ \frac{\rho_1 \rho_2^2}{(\rho_1 + \rho_2)^2} (u_1 - u_2)^2 \right], \quad [9]$$

$$I_{21} = \frac{\rho_2}{\bar{\rho}_2} \bar{I}_{21} = \frac{\rho_1}{\rho_1 + \rho_2} (u_2 - u_1) \Gamma + \frac{\partial}{\partial x} \left[ \frac{\rho_1^2 \rho_2}{(\rho_1 + \rho_2)^2} (u_1 - u_2)^2 \right] \quad [10]$$

and the viscous interaction forces:

$$V_{12} = (\rho_1 / \bar{\rho}_1) \bar{V}_{12} = \rho_1 F_{12} (u_2 - u_1), \quad [11]$$

$$V_{21} = (\rho_2 / \bar{\rho}_2) \bar{V}_{21} = \rho_2 F_{21} (u_1 - u_2), \quad [12]$$

and

$$\rho_1 F_{12} = \rho_2 F_{21} \quad [13]$$

where  $F_{12}$  is the inverse relaxation time of momentum transfer from phase 2 to phase 1;  $F$  is known for spherical droplets, for instance, but remains to be an empirical constant for many other multiphase regimes. With the above forces, adding [3] to the corresponding relation for phase 2 gives [8].

In [9], the first term on the R.H.S. represents the inertial effect: when a given mass in phase 2 is converted to phase 1, its velocity must be brought from  $u_2$  to  $u_1$ . The second term is the inertial force arising from the continuity of the phase boundary, or the force due to reduced or virtual mass of species 2 acting on species 1. This virtual mass might be referred to as a thermodynamic or continuum virtual mass of a two-phase mixture. Derivation from hydrodynamics on specific configuration such as a sphere 2 in a fluid 1 (Basset 1887) gives  $\frac{1}{2}\rho_1(d/dt)(u_1 - u_2)$  per unit volume of the sphere. Note that  $I$ 's vanish when the species are not converted and for small relative motion.

Different from [1] and [2], [3] and [4] are applicable to an identifiable regime of phase 1 mingled with phase 2. Application of [3] and [4] to a monodispersed system is directly conceivable. Their application to slugs and layers calls for averaging over a representative regime or going back to [1] and [2] with proper boundary conditions, but [9] and [10] will not be directly applicable.

Note that, when  $\rho_2 \gg \rho_1$ , such as in a steam (1)–water (2) mixture with a small volume fraction of bubbles, we have

$$I_{12} \approx -(u_1 - u_2)\Gamma + \frac{\partial}{\partial x}[\rho_1(u_1 - u_2)^2]$$

$$I_{21} \approx \frac{\rho_1}{\rho_2}(u_1 - u_2)\Gamma + \frac{\partial}{\partial x}\left[\frac{\rho_1^2}{\rho_2}(u_1 - u_2)^2\right]$$

that is, the inertial effect has a greater influence on steam than on water and  $I_{21} \ll I_{12}$ ;  $I_{21}$  is then negligible. However, in the case of a dilute suspension of mist,  $\rho_2 \ll \rho_1$ , the reverse of the above is true.

The sum of [3] and that for phase 2 can be expressed as

$$\rho_1 \frac{du_1}{dt_1} + \rho_2 \frac{du_2}{dt_2} + \Gamma(u_1 - u_2) - \frac{\partial}{\partial x} \left[ \frac{\rho_1 \rho_2}{\rho_1 + \rho_2} (u_1 - u_2)^2 \right] = - \frac{\partial P}{\partial x} = \rho_m \frac{du_m}{dt_m} \quad [14]$$

The third and fourth terms on the L.H.S. were neglected in many earlier studies, but they are not negligible when there is phase change and when the relative velocity between the phases is large. Alternately, the momentum equation of each phase can be expressed in the form

$$\frac{\partial \rho_1 u_1}{\partial t} + \frac{\partial \rho_1 u_1^2}{\partial x} - \frac{\rho_1 u_1 + \rho_2 u_2}{\rho_1 + \rho_2} \Gamma - \frac{\partial}{\partial x} \left[ \frac{\rho_1 \rho_2}{(\rho_1 + \rho_2)^2} (u_1 - u_2)^2 \right] = - \frac{\rho_1}{\bar{\rho}_1} \frac{\partial P}{\partial x} + \rho_1 F_{12}(u_2 - u_1). \quad [15]$$

Adding [15] to a corresponding form for phase 2 gives

$$\frac{\partial}{\partial t}(\rho_1 u_1 + \rho_2 u_2) + \frac{\partial}{\partial x}(\rho_1 u_1^2 + \rho_2 u_2^2) - \frac{\partial}{\partial x} \left[ \frac{\rho_1 \rho_2}{\rho_1 + \rho_2} (u_1 - u_2)^2 \right] = - \frac{\partial P}{\partial x}. \quad [16]$$

The third term on the L.H.S. arises again because of the change of frame of reference. Note that the quantities under the derivative signs in the second and third terms on the L.H.S. combine to give  $(\rho_1 u_1 + \rho_2 u_2)^2 / (\rho_1 + \rho_2)$ ; hence, [16] reduces to

$$\frac{\partial \rho_m u_m}{\partial t} + \frac{\partial \rho_m u_m^2}{\partial x} = - \frac{\partial P}{\partial x} \quad [17]$$

which can also be obtained by combining [7] and [8].

#### ENERGY EQUATIONS

Excluding heat source and dissipation in our example, the energy equation of adiabatic flow of the mixture can be expressed as

$$\rho_m \frac{dE_m}{dt_m} = \rho_m \frac{\partial E_m}{\partial t} + \rho_m u_m \frac{\partial E_m}{\partial x} = - \frac{\partial u_m P}{\partial x} \quad [18]$$

where the R.H.S. gives the work done by the displacement under pressure  $P$ , the energy  $E$  represents  $U + u^2/2$ ,  $U$  being the internal energy, and  $\rho_m E_m = \rho_1 E_1 + \rho_2 E_2$ . The energy equation of phase 1 in the mixture is now

$$\rho_1 \frac{dE_1}{dt_1} = -\phi_1 \frac{\partial u_1 P}{\partial x} + \phi_1 \frac{\partial}{\partial x} \left[ \frac{\rho_2}{\rho_1 + \rho_2} (u_1 - u_2) P \right] - \Gamma E_1 + \frac{\partial}{\partial x} \left[ \frac{\rho_1 \rho_2}{\rho_1 + \rho_2} (u_1 - u_2) E_1 \right] + c_1 \rho_1 G_{12} (T_2 - T_1) \quad [19]$$

where  $G_{12}$  is the relaxation time for energy transfer from phase 2 to phase 1, and  $c_1 \rho_1 G_{12} = c_2 \rho_2 G_{21}$ ,  $c$  is the specific heat and  $T$  is the temperature. On the R.H.S. of [19], the first term is the rate of work done by phase 1 per unit volume, the second term is that done by phase 2 on phase 1, the third term is the energy removed by phase change, the fourth is the energy input by relative motion, and the last term is the energy transferred by heat flow. The second and third terms constitute the virtual mass effect of species 2 on species 1. Hence, these terms are counterparts of the terms in [3] for momentum transfer. The intraphase energy equation is correspondingly given by  $\bar{\rho}_1 dE_1/dt_1$ . Note again that the work done by phase 1 is by pressure  $P$  instead of its partial pressure  $P_1$ .

Adding [19] to a corresponding relation for phase 2 gives

$$\rho_1 \frac{dE_1}{dt_1} + \rho_2 \frac{dE_2}{dt_2} + \Gamma (E_1 - E_2) - \frac{\partial}{\partial x} \left[ \frac{\rho_1 \rho_2}{\rho_1 + \rho_2} (u_1 - u_2) (E_1 - E_2) \right] = - \frac{\partial u_m P}{\partial x} \quad [20]$$

showing the difference between  $\rho_1(dE_1/dt_1) + \rho_2(dE_2/dt_2)$  and  $\rho_m(dE_m/dt_m)$ . Combining [20] with the continuity equation, [15], and that for phase 2 gives

$$\begin{aligned} \frac{\partial}{\partial t} (\rho_1 E_1 + \rho_2 E_2) + \frac{\partial}{\partial x} \left[ (\rho_1 u_1 E_1 + \rho_2 u_2 E_2) - \frac{\rho_1 \rho_2}{\rho_1 + \rho_2} (u_1 - u_2) (E_1 - E_2) \right] \\ = \frac{\partial (\rho_m E_m)}{\partial t} + \frac{\partial}{\partial x} (\rho_m u_m E_m) = - \frac{\partial u_m P}{\partial x} \end{aligned} \quad [21]$$

which is also given by combining [18] with the continuity equation.

This exploration of intra- and interphase relations was made because of an interest in the nature of the equations of phases of a monodispersed system which appear to be improperly posed (Gidaspow 1974). The equations become well-posed when the effect of force due to pressure gradient is counteracted by the change in the viscous force on a sphere due to acceleration of the fluid (Soo 1975). This is not surprising because the virtual mass is also affected by the viscosity of the fluid (Basset 1887).

The above relations are applicable to suspensions of identifiable average characteristics or a quasi-continuum regime. Even though the intraphase relations are applicable to a slug or a layer, the above inertial interaction force and the viscous interaction force of continuum are not valid and have to be replaced by specific boundary conditions.

#### REFERENCES

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